“Realizing Strategies for winning games”

Senior Project

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Math 498

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Outline of Project

- Briefly show how math relates to popular board games in playing surfaces & strategies as well as teaching mathematical concepts
- Analyzing the game Lights Out by use of linear algebra
- Show how games can help students learn mathematical concepts

•Math is seen related to popular board games

- Chess

Math is seen related to popular board games

- Monopoly

Math is seen related to popular board games

- Scrabble

Math is seen related to popular board games

- Clue

Math is seen related to popular board games

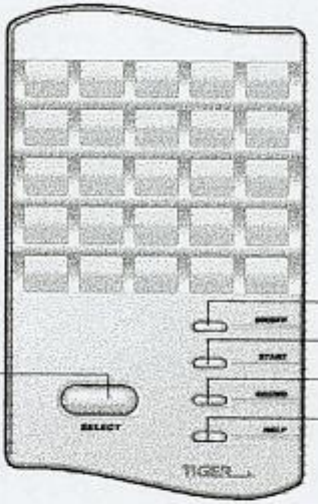
- Connect Four

Mathematical skills learned by board games

Chess	Monopoly	Scrabble	Clue	Connect Four
Memory	Money Skills	Decision Making	Logical & deductive reasoning	Visual perceptual Organization Skills
Game theoretic ideas	Doubling Skills	Reinforcement of the learning of mathematical operations	Predicting & Planning	Helping to learn to read & build charts & graphs & to align columns
Logic	Logical & deductive reasoning	Probability	Visual perceptual skills	
Reasoning & Problem Solving Ability	Probability	Strategic Thinking Skills	Organizational Skills	
Concentration & visualization skills			Problem Solving Skills	

Introduction of Lights Out

- The game itself:


THE GAME

Before you get started, you may want to become accustomed to the different buttons and features on your LIGHTS OUT game.

How to play Lights Out

- Configuration of lights appears
- Pushing a single button will change the on/off state of the light pushed and the adjacent buttons to that button
- Make a series of moves that will turn all the lights out
- Proceed to next puzzle

Linear Algebra Terms to Review

- Gauss-Jordan: applying elementary row operations to a matrix to obtain a reduced row-echelon form
- column space: the subspace of R^n spanned by the column vectors of an m x n matrix
- column vector: a matrix that has only one column

Linear Algebra Terms to Review

- transpose: the matrix obtained by placing the columns of a given matrix into rows, with the first column becoming the first row, etc.
- null space: the solution space of the syste $A\vec{x} = \vec{0}$
- rank: the dimension of the row space (and the column space) of a matrix

Linear Algebra Terms to Review

- free variable: one that can take on any real value
- orthogonal: a property of two vectors in an inner product space stating that their inner product is zero
- basis: a finite set of vectors which is linearly independent and spans a vector space

Mathematical analysis of a winning strategy to Lights Out

- Developed by the use of linear algebra
- 2 initial observations of the game:
  - pushing a button twice is the same as not pushing it at all
  - the on/off state of a button depends on how often (whether even or odd) it and its neighbors have been pushed; the order in which the buttons are pushed does not matter

Mathematical analysis of a winning strategy to Lights Out

- Z_2 represents the use of modulo 2 arithmetic which is the use of only 2 numbers which are 1 and 0
- Examples of modulo 2 addition:
  1+1=0
  1+0=1
  0+1=1

Mathematical analysis of a winning strategy to Lights Out

- The entire array is represented by a 25 x 1 column vector *b*; the state of each light $= b_{i,j}$
- Pressing a single button changes the pattern of lights by adding to *b* a vector that has 1's at the location of the button and its neighbors and 0's elsewhere
- A strategy is represented by another 25 x 1 column vector *x*, where $x_{i,j}$ is 1 is the (i,j) button is to be pushed, and 0 otherwise

Vector *b*, Vector *x* & Obtaining configuration *b* by strategy *x*

$$\vec{b} = (b_{1,1}, b_{1,2}, \ldots, b_{1,5}, b_{2,1}, \ldots b_{5,5})^T$$

- Both vectors are 25 x 1 column vectors

$$\vec{x} = (x_{1,1}, x_{1,2}, \ldots, x_{1,5}, x_{2,1}, \ldots, x_{5,5})^T$$

- Starting with all the lights out, then:

$$b_{1,1} = x_{1,1} + x_{1,2} + x_{2,1}$$
$$b_{1,2} = x_{1,1} + x_{1,2} + x_{1,3} + x_{2,2}$$
$$b_{1,3} = x_{1,2} + x_{1,3} + x_{1,4} + x_{2,3}$$

Checking the result

- The matrix product $A\vec{x} = \vec{b}$ checks that the result $\vec{b}$ is of the strategy $\vec{x}$, with matrix A defined in the next slide.
- Given a puzzle $\vec{b}$, it is winnable if there exists a strategy $\vec{x}$ to turn out all the lights in $\vec{b}$.
- To find a strategy, solve $\vec{b} = A\vec{x}$

Matrix A: 25 x 25

- $$\begin{pmatrix} B & I & O & O & O \\ I & B & I & O & O \\ O & I & B & I & O \\ O & O & I & B & I \\ O & O & O & I & B \end{pmatrix}$$

- I = the 5 x5 identity matrix
- 0 = the 5 x 5 matrix of all zeros
- B = the 5 x 5 matrix shown next

Matrix B:

- $$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

- Matrix A and Matrix B are both symmetric

Mathematica will find the column space of matrix A

- Perform Gauss-Jordan elimination on A
- Use commands RowReduce and Mod inside of Mathematica to analyze matrix A
- Gauss-Jordan will yield RA=E
- E=the Gauss-Jordan echelon form
- R= the product of the elementary matrices which perform the row reducing operations
- A=the 25 x 25 matrix defined previously and shown in full on next slide

Finding the column space of A

- **This is Matrix A:**

A = Mod[

RowReduce[
$$\begin{pmatrix}
1&1&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
1&1&1&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&1&1&1&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&1&1&1&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
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0&0&0&0&1&0&0&0&1&1&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&1&0&0&0&0&1&1&0&0&0&1&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&1&0&0&0&1&1&1&0&0&0&1&0&0&0&0&0&0&0&0\\
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0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&1&1&0&0&0&1&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&1&1&1&0&0&0&1&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&1&1&1&0&0&0&1&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&1&1&1&0&0&0&1&0\\
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0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&1&1&1&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&1&1&1\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&1&1
\end{pmatrix}$$
],

2]

- **Matrix A reduced:**

$$\begin{pmatrix}
1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1\\
0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0\\
0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1\\
0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0\\
0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1\\
0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1\\
0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1\\
0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1\\
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0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&1&0\\
0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&1&0\\
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0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&1&1\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&1&1\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&1\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&1&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&1\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0
\end{pmatrix}$$

Analyzing the column space of A

- Matrix E is of rank 23, with two free variables: $x_{5,4}$ and $x_{5,5}$
- The last two columns of E are:

$$(0,1,1,1,0,1,0,1,0,1,1,1,0,1,1,1,0,1,0,1,0,1,1,0,0)^T$$

and

$$(1,0,1,0,1,1,0,1,0,1,0,0,0,0,0,1,0,1,0,1,1,0,1,0,0)^T$$

- Since A is symmetric, the column space of A=the row space of A

Analyzing the column space of A

- The row space of A is the orthogonal complement of the null space of A, which in turn equals the null space of E
- To describe the column space of A, we need to determine a basis for the null space of E
- Examine the last 2 columns of E which are:

$$\vec{n}_1 = (0,1,1,1,0,1,0,1,0,1,1,1,0,1,1,1,0,1,0,1,0,1,1,0,0)^T$$

$$\vec{n}_2 = (1,0,1,0,1,1,0,1,0,1,0,0,0,0,0,1,0,1,0,1,1,0,1,0,0)^T$$

Theorems for solutions

- **Theorem 1.** ***A configuration $\vec{b}$ is winnable if and only if $\vec{b}$ is perpendicular to the two vectors $\vec{n}_1 \text{ and } \vec{n}_2$.***
- **Theorem 2.** ***Suppose that $\vec{b}$ is a winnable configuration. Then the four winning strategies for $\vec{b}$ are:***

$$R\vec{b},\ R\vec{b}+\vec{n}_1,\ R\vec{b}+\vec{n}_2,\ R\vec{b}+\vec{n}_1+\vec{n}_2$$

Practical method of solving puzzles in Lights Out

- For every on light in the top row, press the button under it to turn it off.
- Repeat step one for rows 2,3,4.
- If the bottom row is all off, you are done. If the bottom row has any of the following patterns, the puzzle can be solved:

00111	01010	01101
10001	10110	11011
11100		

Practical method of solving puzzles in Lights Out

- The puzzle cannot be solved if any other configuration is left on row 5
- To actually solve these puzzles, number the buttons in the top row from left to right 1,2,3,4,5. Find the pattern in row 5 in the following table and press the top row button(s) indicated:

Practical method of solving puzzles in Lights Out

00111	4
01010	1 & 4
01101	1
10001	1 & 2
10110	5
11011	3
11100	2

Linking the linear algebra method to the practical method

- Any of the previous configurations shown are orthogonal to both vectors $\vec{n}_1$ and $\vec{n}_2$
- Example:
- (0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,1,0,0,0,1) dotted with $\vec{n}_1$ yields 0 and

(0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,1,0,0,0,1) dotted with $\vec{n}_2$ also yields 0

Games used in the classroom

Advantages

- solidify mathematical reasoning & calculating skills
- development of strong logical thinking skills and fine motor skills
- Other thinking skills that develop are: interpretation, optimization, analysis, variation, probability, and generalization

Disadvantages

- students move chairs & tables and circulate freely which can disrupt the class room
- students gather in groups and argue strategy when playing a game
- divert from the conventional classroom teachings

Characteristics of ‘mathematical games’

- only 2 players
- involve only thinking skills
- offer full information at all times
- do not, in general, involve luck
- usually are finished within a reasonable span of time
- are also played for pleasure
- require a minimum of special equipment

Examples of mathematical games

- Noughts & Crosses (Tic-Tac-Toe)
- Nim
- Make 15
- Blox
- End to End
- Odd Wins
- Triangle Sum
- Die Adds
- Capture the Numbers
- Diox
- Winners or Losers

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